

BMO-TYPE NORMS RELATED TO THE PERIMETER OF SETS

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ABSTRACT. In this paper we consider an isotropic variant of the *BMO*-type norm recently introduced in [6]. We prove that, when considering characteristic functions of sets, this norm is related to the perimeter. A byproduct of our analysis is a new characterization of the perimeter of sets in terms of this norm, independent of the theory of distributions.

1. INTRODUCTION

Let $Q = (0, 1)^n$ be the unit cube in \mathbb{R}^n , $n > 1$. In a very recent paper [6], the second and third author, in collaboration with P. Mironescu, introduced a new function space $B \subset L^1(Q)$ based on the following seminorm, inspired by the celebrated *BMO* space of John-Nirenberg [14]:

$$\|f\|_B := \sup_{\epsilon \in (0,1)} [f]_\epsilon,$$

where

$$[f]_\epsilon := \epsilon^{n-1} \sup_{\mathcal{G}_\epsilon} \sum_{Q' \in \mathcal{G}_\epsilon} \int_{Q'} |f(x) - \int_{Q'} f| dx, \quad (1.1)$$

and \mathcal{G}_ϵ denotes a collection of disjoint ϵ -cubes $Q' \subset Q$ with sides parallel to the coordinate axes and cardinality not exceeding ϵ^{1-n} ; the supremum in (1.1) is taken over all such collections. In addition to $\|f\|_B$, it is also useful to consider its infinitesimal version, namely

$$[f] := \limsup_{\epsilon \downarrow 0} [f]_\epsilon \leq \|f\|_B,$$

and the space $B_0 := \{f \in B : [f] = 0\}$.

Their main motivation was the search of a space X on the one hand sufficiently large to include *VMO*, $W^{1,1}$, and the fractional Sobolev spaces $W^{1/p,p}$ with $1 < p < \infty$, on the other hand sufficiently small (i.e., with a sufficiently strong seminorm) to provide the implication

$$f \in X \text{ and } \mathbb{Z}\text{-valued} \quad \implies \quad f = k \text{ } \mathcal{L}^n\text{-a.e. in } Q, \text{ for some } k \in \mathbb{Z} \quad (1.2)$$

an implication known to be true in the spaces *VMO*, $W^{1,1}$, and $W^{1/p,p}$.

One of the main results in [6] asserts that (1.2) holds with $X = B_0$. The principal ingredient in the proof concerns the case $f = \mathbf{1}_A$, where $A \subset Q$ is measurable. For such

special functions it is proved in [6] that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C [f]; \quad (1.3)$$

here and in what follows we denote by C a generic constant depending only on n . Estimate (1.3) suggests a connection with Sobolev embeddings and isoperimetric inequalities; recall e.g. that

$$\left\| f - \fint_Q f \right\|_{L^{n/(n-1)}(Q)} \leq C |Df|(Q) \quad \forall f \in BV(Q). \quad (1.4)$$

When $f = \mathbf{1}_A$, (1.4) takes the form

$$\left\| \mathbf{1}_A - \fint_Q \mathbf{1}_A \right\|_{L^{n/(n-1)}(Q)} \leq C P(A, Q), \quad (1.5)$$

where $P(A, Q)$ denotes the perimeter of A relative to Q . Combining (1.5) with the obvious inequality

$$\left\| \mathbf{1}_A - \fint_Q \mathbf{1}_A \right\|_{L^{n/(n-1)}(Q)} \leq 2$$

yields

$$\left\| \mathbf{1}_A - \fint_Q \mathbf{1}_A \right\|_{L^{n/(n-1)}(Q)} \leq C \min\{1, P(A, Q)\}. \quad (1.6)$$

In view of (1.3) and (1.6) it is natural to ask whether there exists a relationship between $[\mathbf{1}_A]$ and $\min\{1, P(A, Q)\}$. The aim of this paper is to answer positively to this question.

Since the concept of perimeter is isotropic, it is better to make also the main object of [6] isotropic, by considering

$$l_\epsilon(f) := \epsilon^{n-1} \sup_{\mathcal{F}_\epsilon} \sum_{Q' \in \mathcal{F}_\epsilon} \fint_{Q'} |f(x) - \fint_{Q'} f| dx, \quad (1.7)$$

where \mathcal{F}_ϵ denotes a collection of disjoint ϵ -cubes $Q' \subset \mathbb{R}^n$ with *arbitrary* orientation and cardinality not exceeding ϵ^{1-n} .

The main result of this paper is the following:

Theorem 1.1. *For any measurable set $A \subset \mathbb{R}^n$ one has*

$$\lim_{\epsilon \rightarrow 0} l_\epsilon(\mathbf{1}_A) = \frac{1}{2} \min\{1, P(A)\}. \quad (1.8)$$

In particular, $\lim_{\epsilon} l_\epsilon(\mathbf{1}_A) < 1/2$ implies that A has finite perimeter and $P(A) = 2 \lim_{\epsilon} l_\epsilon(\mathbf{1}_A)$.

We present the proof of Theorem 1.1 in Section 3. Although we confine ourselves to the most interesting case $n > 1$ throughout this paper, we point out in Section 3.4 that Theorem 1.1 still holds in the case $n = 1$ and we present a brief proof; since in this case $P(A)$ is an integer, we infer that $\lim_{\epsilon} l_\epsilon(\mathbf{1}_A) < 1/2$ implies that either A or $\mathbb{R} \setminus A$ are Lebesgue negligible.

Returning to the case $n > 1$, we can also understand better the role of the upper bound on cardinality with the formula

$$\limsup_{\epsilon \downarrow 0} \epsilon^{n-1} \sum_{Q' \in \mathcal{F}_{\epsilon, M}} 2 \int_{Q'} |\mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A| dx = \min\{M, P(A)\} \quad \forall M > 0, \quad (1.9)$$

where $\mathcal{F}_{\epsilon, M}$ denotes a collection of ϵ -cubes with arbitrary orientation and cardinality not exceeding $M\epsilon^{1-n}$. The proof of (1.9) can be achieved by a scaling argument. Indeed, setting $\rho = M^{-1/(n-1)}$, it suffices to apply (1.8) to $\tilde{A} = \rho A$, noticing that

$$\min\{1, P(\tilde{A})\} = \frac{1}{M} \min\{M, P(A)\} = \rho^{n-1} \min\{M, P(A)\},$$

that $\epsilon^{1-n} = M(\epsilon/\rho)^{1-n}$, and finally that the transformation $x \mapsto x/\rho$ maps \tilde{A} to A , as well as ϵ -cubes to ϵ/ρ -cubes.

In Section 4.1 we return to the framework of measurable subsets A of a Lipschitz domain Ω , and we establish that

$$\lim_{\epsilon \rightarrow 0} \mathbf{l}_\epsilon(\mathbf{1}_A, \Omega) = \frac{1}{2} \min\{1, P(A, \Omega)\}, \quad (1.10)$$

where $\mathbf{l}_\epsilon(\mathbf{1}_A, \Omega)$ is a localized version of (1.7) where we restrict the supremum over cubes contained in Ω .

Also, going back to the setting of [6], in Section 4.1 we prove that

$$[\mathbf{1}_A] \leq \frac{1}{2} \min\{1, P(A, Q)\} \leq C [\mathbf{1}_A]$$

for every measurable subset of Q .

Then, in Section 4.2 we discuss how removing the bound on the cardinality allows us to obtain a new characterization both of sets of finite perimeter and of the perimeter, independent of the theory of distributions. In a somewhat different direction, see also [7], [8, Corollary 3 and Equation (46)], and [9].

We conclude this introduction with a few more words on the strategy of proof. As illustrated in Remark 3.1, using the canonical decomposition in cubes, it is not too difficult to show the existence of dimensional constants $\xi_n, \eta_n > 0$ satisfying

$$\limsup_{\epsilon \downarrow 0} \mathbf{l}_\epsilon(\mathbf{1}_A) < \xi_n \quad \implies \quad P(A) \leq \eta_n \limsup_{\epsilon \downarrow 0} \mathbf{l}_\epsilon(\mathbf{1}_A) \quad (1.11)$$

for any measurable set $A \Subset Q$. This idea can be very much refined, leading to the proof of the inequality $\liminf_{\epsilon} \mathbf{l}_\epsilon(\mathbf{1}_A) \geq 1/2$ whenever $P(A) = +\infty$. Since $\mathbf{l}_\epsilon(\mathbf{1}_A) \leq 1/2$, see (3.1) below, this proves our main result for sets of infinite perimeter. For sets of finite perimeter, the inequality \leq in (1.8) relies on the relative isoperimetric inequality in the cube with sharp constant, see (2.2) below, while the inequality \geq relies on a blow-up argument.

This paper originated from a meeting in Naples in November 2013, dedicated to Carlo Sbordone's 65th birthday, where three of us (LA, HB, and AF) met. On that occasion HB presented some results from [6] and formulated a conjecture which became the motivation

and the main result of the present paper. For this and many other reasons, we are happy to dedicate this paper to Carlo Sbordone.

2. NOTATION AND PRELIMINARY RESULTS

Throughout this paper we assume $n \geq 2$. We denote by $\# F$ the cardinality of a set F , by A^c the complement of A , by $|A|$ the Lebesgue measure of a (Lebesgue) measurable set $A \subset \mathbb{R}^n$, by \mathcal{H}^{n-1} the Hausdorff $(n-1)$ -dimensional measure. For $\delta > 0$, we say that Q' is a δ -cube if Q' is a cube obtained by rotating and translating the standard δ -cube $(0, \delta)^n$.

2.1. BV functions, sets of finite perimeter, and relative isoperimetric inequalities. Given $\Omega \subset \mathbb{R}^n$ open and $f \in L^1_{\text{loc}}(\Omega)$, we define

$$|Df|(\Omega) := \sup \left\{ \int f(x) \operatorname{div} \phi(x) dx : \phi \in C_c^1(\Omega; \mathbb{R}^n), |\phi| \leq 1 \right\}. \quad (2.1)$$

By construction, $f \mapsto |Df|(\Omega) \in [0, \infty]$ is lower semicontinuous w.r.t. the $L^1_{\text{loc}}(\Omega)$ convergence. By Riesz theorem, whenever $|Df|(\Omega)$ is finite the distributional derivative $Df = (D_1 f, \dots, D_n f)$ of f is a vector-valued measure with finite total variation, therefore $f \in BV_{\text{loc}}(\Omega)$.¹ In addition, the total variation of Df coincides with the supremum in (2.1) (thus, justifying our notation).

We will mostly apply these concepts when $f = \mathbf{1}_A$ is a characteristic function of a measurable set $A \subset \mathbb{R}^n$. In this case we use the traditional and more convenient notation

$$P(A, \Omega) = |D\mathbf{1}_A|(\Omega), \quad P(A) = P(A, \mathbb{R}^n).$$

A key property of the perimeter is the so-called relative isoperimetric inequality: for any bounded open set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary one has

$$|E| \cdot |\Omega \setminus E| \leq c(\Omega) P(E, \Omega) \quad \text{for any measurable set } E \subset \Omega.$$

In the case when Ω is the unit cube Q , we will need the inequality with sharp constant:

$$|E|(1 - |E|) \leq \frac{1}{4} P(E, Q) \quad \text{for any measurable set } E \subset Q. \quad (2.2)$$

This inequality is originally due to H. Hadwiger [13] for polyhedral subsets of the cube. Far reaching variants appeared subsequently in the literature (see e.g. S.G. Bobkov [4, 5], D. Bakry and M. Ledoux [2], F. Barthe and B. Maurey [3], and their references). However we could not find (2.2) stated in the required generality used here (it is often formulated with the Minkowski content instead of the perimeter, so that some extra approximation argument is anyhow needed). For this reason, and for the reader's convenience, we have included in the appendix a proof of (2.2) based on the results of [3], in any number of space dimensions.

¹Recall that $f \in BV(U)$ if $f \in L^1(U)$ and $|Df|(U) < \infty$.

2.2. Fine properties of sets of finite perimeter. In §3.3 we will need finer properties of sets of finite perimeter A in an open set Ω . In [10], De Giorgi singled out a set $\mathcal{F}A$ of finite \mathcal{H}^{n-1} -measure, called reduced boundary, on which $|D\mathbf{1}_A|$ is concentrated and A is asymptotically close to a half-space. More precisely $|D\mathbf{1}_A| = \mathcal{H}^{n-1} \llcorner \mathcal{F}A$, i.e., $|D\mathbf{1}_A|(E) = \mathcal{H}^{n-1}(E \cap \mathcal{F}A)$ for any Borel set $E \subset \Omega$. De Giorgi also proved that $|D\mathbf{1}_A|$ -almost all of the reduced boundary can be covered by a sequence of C^1 hypersurfaces Γ_i (the so-called rectifiability property). A few years later, Federer in [11] extended these results to the so-called essential boundary, namely the complement of density 0 and density 1 sets:

$$\partial^* A := \left\{ x \in \mathbb{R}^n : \limsup_{r \rightarrow 0^+} \frac{|B_r(x) \cap A|}{|B_r(x)|} > 0 \quad \text{and} \quad \limsup_{r \rightarrow 0^+} \frac{|B_r(x) \setminus A|}{|B_r(x)|} > 0 \right\}. \quad (2.3)$$

Federer also slightly strenghtned the rectifiability result, by replacing $|D\mathbf{1}_A|$ with \mathcal{H}^{n-1} . We collect in the next theorem the results we need on sets of finite perimeter.

Theorem 2.1 (De Giorgi-Federer). *Let A be a set of finite perimeter in Ω . Then the following properties hold:*

(i)

$$|D\mathbf{1}_A|(E) = \mathcal{H}^{n-1}(E \cap \partial^* A) \quad \text{for any Borel set } E \subset \Omega; \quad (2.4)$$

(ii) *there exist embedded C^1 hypersurfaces $\Gamma_i \subset \mathbb{R}^n$ satisfying*

$$\mathcal{H}^{n-1} \left(\Omega \cap \partial^* A \setminus \bigcup_{i=1}^{\infty} \Gamma_i \right) = 0; \quad (2.5)$$

- (iii) *if Γ_i are as in (2.5), for \mathcal{H}^{n-1} -a.e. $x \in \Omega \cap \partial^* A \cap \Gamma_i$ there exists a half-space $H_A(x)$ with inner normal $\nu_A(x)$ orthogonal to Γ_i at x such that $\mathbf{1}_{(A-x)/r} \rightarrow \mathbf{1}_{H_A(x)}$ in $L^1_{\text{loc}}(\mathbb{R}^n)$ as $r \rightarrow 0^+$;*
- (iv) *if F is any other set with finite perimeter in Ω , $H_A(x) = \pm H_F(x)$ \mathcal{H}^{n-1} -a.e. in $\Omega \cap \partial^* A \cap \partial^* F$.*

Proof. For the first three properties, see [10], [11], or [1, Theorems 3.59 and 3.61]. Taking (iii) into account, the last assertion (iv) follows from the elementary property

$$\text{Tan}(\Gamma, x) = \text{Tan}(\tilde{\Gamma}, x) \quad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Gamma \cap \tilde{\Gamma}$$

whenever Γ and $\tilde{\Gamma}$ are C^1 embedded hypersurfaces. □

Since $BV_{\text{loc}}(\Omega) \cap L^\infty(\Omega)$ is easily seen to be an algebra with

$$|D(uv)| \leq \|u\|_\infty |Du| + \|v\|_\infty |Dv|,$$

it turns out that the class of sets of finite perimeter in an open set Ω is stable under relative complement, union, and intersection. We need also the following property:

$$\mathcal{H}^{n-1}(\Omega \cap \partial^*(E \Delta F) \setminus (\partial^* E \Delta \partial^* F)) = 0 \quad \text{whenever } E, F \text{ have finite perimeter in } \Omega. \quad (2.6)$$

In order to prove it, we first notice that ∂^* is invariant under complement and $\partial^*(E \cup F) \subset \partial^*E \cup \partial^*F$, $\partial^*(E \cap F) \subset \partial^*E \cup \partial^*F$, hence it follows that $\partial^*(E \Delta F) \subset \partial^*E \cup \partial^*F$. Then, take $x \in \Omega \cap \partial^*(E \Delta F)$ and assume (possibly permuting E and F) that $x \in \partial^*E$. By property (iii) of Theorem 2.1, possibly ignoring a \mathcal{H}^{n-1} -negligible set, we can also assume that $(E - x)/r$ converges as $r \rightarrow 0^+$ to a half-space $H_E(x)$. Still ignoring another \mathcal{H}^{n-1} -negligible set, we have then three possibilities for F : either x is a point of density 1, or a point of density 0, or there exists a half-space $H_F(x)$ such that $(F - x)/r \rightarrow H_F(x)$ as $r \rightarrow 0^+$. In the first two cases it is clear that $x \in \partial^*E \setminus \partial^*F$ and we are done. In the third case, we know by property (iv) of Theorem 2.1 that $H_E(x) = \pm H_F(x)$ for \mathcal{H}^{n-1} -a.e. $x \in \Omega \cap \partial^*E \cap \partial^*F$. But $H_E(x) = H_F(x)$ implies that x is a point of density 0 for $E \Delta F$ and $H_E(x) = -H_F(x)$ implies that x is a point of density 1 for $E \Delta F$, so the third case can occur only on a \mathcal{H}^{n-1} -negligible set.

3. PROOF OF THEOREM 1.1

The proof of Theorem 1.1 is quite involved and will take all of this section. Notice that since

$$\int_{Q'} |\mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A| dx = 2 \frac{|Q' \cap A| \cdot |Q' \setminus A|}{|Q'|^2} \leq \frac{1}{2} \quad (3.1)$$

for any ϵ -cube Q' , we clearly have $\mathbf{l}_\epsilon(\mathbf{1}_A) \leq 1/2$.

We now prove the theorem in three steps: first we show that $\mathbf{l}_\epsilon(\mathbf{1}_A) \leq \mathbf{P}(A)/2$ for all $\epsilon > 0$, which proves that $\mathbf{l}_\epsilon(f) \leq \frac{1}{2} \min\{1, \mathbf{P}(A)\}$. Then we prove that $\liminf_{\epsilon \rightarrow 0^+} \mathbf{l}_\epsilon(f) \geq \frac{1}{2} \min\{1, \mathbf{P}(A)\}$ first when $\mathbf{P}(A) = \infty$ (the non-rectifiable case) and finally when A has finite perimeter (the rectifiable case).

3.1. Upper bound. We prove that $\mathbf{l}_\epsilon(\mathbf{1}_A) \leq \mathbf{P}(A)/2$ for all $\epsilon > 0$. For this, we may obviously assume $\mathbf{P}(A) < \infty$, hence $f = \mathbf{1}_A \in BV_{\text{loc}}(\mathbb{R}^n)$. By the additivity of $\mathbf{P}(A, \cdot)$, it suffices to show that if Q' is an ϵ -cube, then

$$\frac{|Q' \cap A| \cdot |Q' \setminus A|}{|Q'|^2} \leq \frac{1}{4} \epsilon^{1-n} \mathbf{P}(A, Q').$$

After rescaling, this inequality reduces to (2.2), which proves the desired result.

3.2. Lower bound: the non-rectifiable case. Here we assume that $\mathbf{P}(A) = \infty$ and we prove, under this assumption, that $\liminf_{\epsilon} \mathbf{l}_\epsilon(f) \geq 1/2$. Before coming to the actual proof we sketch in the next remark the proof of (1.11), announced in the introduction.

Remark 3.1. Let us consider the canonical subdivision (up to a Lebesgue negligible set) of $(0, 1)^n$ in 2^{hn} cubes $Q_{i,h}$ with length side 2^{-h} . We define on the scale $\epsilon = 2^{-h}$ an approximate interior $\text{Int}_h(A)$ of A by considering the set

$$I_h := \left\{ i \in \{1, \dots, 2^{hn}\} : \int_{Q_{i,h}} \mathbf{1}_A > \frac{3}{4} \right\}$$

and taking the union of the cubes $Q_{i,h}$, $i \in I_h$. Analogously we define a set of indices E_h and the corresponding approximate exterior $\text{Ext}_h(A) = \text{Int}_h(Q \setminus A)$. We denote by F_h the complement of $I_h \cup E_h$ and by $\text{Bdry}_h(Q)$ the union of the corresponding cubes.

Since $\text{Int}_h(A) \rightarrow A$ in L^1_{loc} as $h \rightarrow \infty$, by the lower semicontinuity of the perimeter it suffices to give a uniform estimate on $\mathbf{P}(\text{Int}_h(A)) = \mathcal{H}^{n-1}(\partial \text{Int}_h(A))$ as $h \rightarrow \infty$ under a smallness assumption on $\limsup_h \mathbf{l}_{2^{-h}}(\mathbf{1}_A)$.

Since $\int_{Q_{i,h}} |\mathbf{1}_A(x) - \int_{Q_{i,h}} \mathbf{1}_A| dx \geq 1/4$ for all $i \in F_h$ (by definition of F_h), we obtain that

$$\mathbf{l}_{2^{-h}}(\mathbf{1}_A) < \frac{1}{4} \quad \implies \quad \# F_h \leq 4 \mathbf{l}_{2^{-h}}(\mathbf{1}_A) (2^{-h})^{1-n} < (2^{-h})^{1-n}, \quad (3.2)$$

which provides a uniform estimate on $\mathcal{H}^{n-1}(\partial \text{Bdry}_h(A))$. Hence, to control $\mathcal{H}^{n-1}(\partial \text{Int}_h(A))$ it suffices to bound the number of faces $F \subset Q$ common to a cube $Q_{i,h}$ and a cube $Q_{j,h}$, with $i \in I_h$ and $j \in E_h$. For this, notice that if \tilde{Q} is any cube with side length 2^{1-h} containing $Q_{i,h} \cup Q_{j,h}$, it is easily seen that

$$\int_{\tilde{Q}} |\mathbf{1}_A(x) - \int_{\tilde{Q}} \mathbf{1}_A| dx \geq 2^{-1-n}$$

and this leads once more to an estimate of the number of these cubes with $(2^{1-h})^{1-n}$ provided $\mathbf{l}_{2^{1-h}}(\mathbf{1}_A) < 2^{-1-n}$. Combining this estimate with the uniform estimate on $\mathcal{H}^{n-1}(\partial \text{Bdry}_h(A))$ leads to (1.11).

We now refine the strategy above to prove:

Lemma 3.2. *Let $K > 0$ and $A \subset \mathbb{R}^n$ measurable with $\mathbf{P}(A) = \infty$. Then there exists $\delta_0 = \delta_0(K, A) > 0$ with the following property: for all $\delta \in (0, \delta_0]$ it is possible to find a disjoint collection \mathcal{U}_δ of δ -cubes satisfying:*

- (a) $2^{-n-1} < |Q' \cap A|/|Q'| < 1 - 2^{-n-1}$ for all $Q' \in \mathcal{U}_\delta$;
- (b) $\#\mathcal{U}_\delta > K\delta^{-n+1}$;
- (c) if $\mathcal{U}_\delta = \{Q_\delta(x_i)\}_{i \in I}$, the homothetic cubes $\{Q_{2\delta}(x_i)\}_{i \in I}$ are pairwise disjoint.

Proof. In this proof we tacitly assume that all cubes have sides parallel to a fixed system of coordinates. Partition canonically \mathbb{R}^n in a family $\{Q_i\}_{i \in \mathbb{Z}^n}$ of $\delta/2$ -cubes and set

$$\mathcal{V}_{\delta/2} := \left\{ Q_i : \frac{|Q_i \cap A|}{|Q_i|} > \frac{1}{2} \right\}, \quad A_{\delta/2} := \bigcup_{Q_i \in \mathcal{V}_{\delta/2}} Q_i.$$

Since $A_{\delta/2} \rightarrow A$ locally in measure as $\delta \rightarrow 0^+$, it follows from the lower semicontinuity of \mathbf{P} that

$$\liminf_{\delta \rightarrow 0^+} \mathbf{P}(A_{\delta/2}) \geq \mathbf{P}(A) = \infty.$$

We define $\delta_0 = \delta_0(K, A) > 0$ by requiring that $\mathbf{P}(A_{\delta/2}) > 2^{2n+2}nK$ for all $\delta \in (0, \delta_0]$.

Fixing now $\delta \in (0, \delta_0]$ and defining

$$\tilde{\mathcal{V}} := \{Q_i \in \mathcal{V}_{\delta/2} : \mathcal{H}^{n-1}(\partial Q_i \cap \partial A_{\delta/2}) > 0\}$$

as the subset of “boundary cubes” (see Figure 1) we can estimate

$$2^{2n+2}nK < P(A_{\delta/2}) \leq 2n\left(\frac{\delta}{2}\right)^{n-1} \# \tilde{\mathcal{V}},$$

so that

$$\# \tilde{\mathcal{V}} > 8^n K \delta^{-n+1}. \quad (3.3)$$

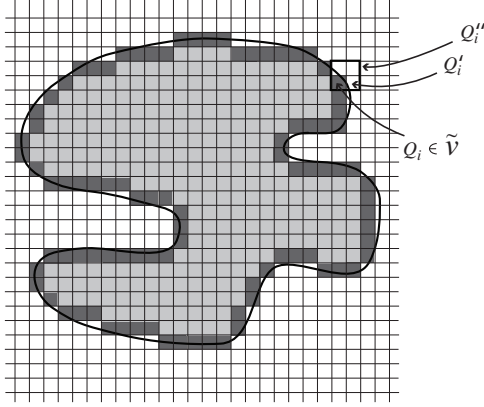


FIGURE 1.

Let $Q_i \in \tilde{\mathcal{V}}$ and let Q'_i be a $\delta/2$ -cube sharing a face $\sigma \subset \partial Q_i \cap \partial A_{\delta/2}$ with Q_i (see Figure 1). Since obviously $Q'_i \notin \mathcal{V}_{\delta/2}$ we obtain $|Q'_i \cap A| \leq |Q'_i|/2$. Hence, if Q''_i is any δ -cube containing $Q_i \cup Q'_i$ we have

$$\begin{cases} |Q''_i \cap A| \geq |Q_i \cap A| > \frac{1}{2}|Q_i| = \frac{1}{2^{n+1}}|Q''_i|, \\ |Q''_i \cap A^c| \geq |Q'_i \cap A^c| \geq \frac{1}{2}|Q'_i| = \frac{1}{2^{n+1}}|Q''_i|. \end{cases}$$

It then suffices to consider a maximal subfamily $\mathcal{V}^* \subset \tilde{\mathcal{V}}$ of $\delta/2$ cubes with centers at mutual distance (along at least one of the coordinate directions) larger or equal than $7\delta/2$ and define

$$\mathcal{U}_\delta := \{Q''_i : Q_i \in \mathcal{V}^*\}.$$

It is easy to check that \mathcal{U}_δ is a family of δ -cubes whose homothetic enlargements by a factor 2 along their centers are disjoint, so that (c) holds, and that (a) holds as well. In order to check (b), we notice that the union of the enlargements by a factor 8 of all cubes in \mathcal{V}^* contains $\tilde{\mathcal{V}}$, by the maximality of \mathcal{V}^* . Hence, from (3.3) we get

$$\# \mathcal{V}^* \geq 8^{-n} \# \tilde{\mathcal{V}} > K \delta^{-n+1}.$$

□

Lemma 3.3. *Let $c_0 \in (0, 1/2)$ and $A \subset (0, 1)^n = Q$ measurable, with*

$$c_0 < |A| < 1 - c_0. \quad (3.4)$$

Then, there exists $\epsilon_0 = \epsilon_0(c_0, A) > 0$ with the following property: for $\epsilon \in (0, \epsilon_0)$ there exists a disjoint collection \mathcal{G}_ϵ of ϵ -cubes contained in $(0, 1)^n$ and satisfying

$$\frac{|V \cap A|}{|V|} = \frac{1}{2} \quad \forall V \in \mathcal{G}_\epsilon, \quad (3.5)$$

$$\#\mathcal{G}_\epsilon > c_1 \epsilon^{-n+1}, \quad (3.6)$$

with $c_1 > 0$ depending only on c_0 .

Proof. First we choose $\epsilon_* = \epsilon_*(c_0, n) \in (0, 1/2)$ such that the sets $A_1 := A \cap (\epsilon_*, 1 - \epsilon_*)^n \subset A$ and $A_2 := A \cup [(0, 1)^n \setminus (\epsilon_*, 1 - \epsilon_*)^n] \supset A$ satisfy

$$\frac{c_0}{2} < |A_1|, \quad |A_2| < 1 - \frac{c_0}{2}. \quad (3.7)$$

We now extend the set A_2 by periodicity:

$$\tilde{A}_2 := \bigcup_{h \in \mathbb{Z}^n} A_2 + h, \quad \tilde{A}_2^c = \mathbb{R}^n \setminus \tilde{A}_2.$$

Then (3.7) implies

$$\int_Q \int_Q \mathbf{1}_{A_1}(x) \mathbf{1}_{\tilde{A}_2^c}(x+z) dx dz = |A_1|(1 - |A_2|) > \frac{c_0^2}{4}.$$

Hence, we can find a nonzero vector $z \in Q$ satisfying

$$|A_1 \cap (\tilde{A}_2^c - z)| > \frac{c_0^2}{4}.$$

Set now $e := z/|z|$, $H_a := z^\perp + ae$, $\hat{A} := A_1 \cap (\tilde{A}_2^c - z)$, and

$$A_\delta := \left\{ x \in \hat{A} : \frac{|Q_r(y) \cap A_1|}{|Q_r(y)|} > \frac{1}{2} \quad \forall y \in Q_r(x), \quad r \in (0, \delta) \right\}.$$

Since A_δ monotonically converge as $\delta \downarrow 0$ to a set containing the set of points of density 1 of \hat{A} , it follows that $|A_\delta| > c_0^2/4$ for δ small enough. Hence, because

$$|A_\delta| \leq 2 \int_{-\sqrt{n}/2}^{\sqrt{n}/2} \mathcal{H}^{n-1}(A_\delta \cap H_a) da,$$

we can find $a \in (-\sqrt{n}/2, \sqrt{n}/2)$ satisfying

$$\mathcal{H}^{n-1}(A_\delta \cap H_a) > \frac{c_0^2}{8\sqrt{n}}. \quad (3.8)$$

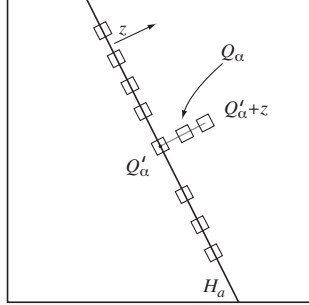


FIGURE 2.

For $\epsilon \leq \delta$, let us consider a canonical division $\{R_\alpha\}_{\alpha \in \mathbb{Z}^{n-1}}$ of H_a in ϵ -cubes of dimension $n-1$, and select those cubes R_α that satisfy $\mathcal{H}^{n-1}(A_\delta \cap R_\alpha) > \mathcal{H}^{n-1}(R_\alpha)/2$, to build a family \mathcal{R}_ϵ . Since

$$\bigcup_{R_\alpha \in \mathcal{R}_\epsilon} R_\alpha \rightarrow A_\delta \cap H_a \quad \text{in } \mathcal{H}^{n-1}\text{-measure as } \epsilon \rightarrow 0^+,$$

we obtain from (3.8)

$$\limsup_{\epsilon \rightarrow 0} \epsilon^{n-1} \# \mathcal{R}_\epsilon > \frac{c_0^2}{8\sqrt{n}}. \quad (3.9)$$

Out of \mathcal{R}_ϵ we can build a disjoint collection \mathcal{G}'_ϵ of ϵ -cubes Q'_α centered at points $x_\alpha \in H_a$ with faces either orthogonal or parallel to z , such that

$$|Q'_\alpha \cap \hat{A}| > \frac{1}{2}|Q'_\alpha|, \quad (3.10)$$

$$\# \mathcal{G}'_\epsilon > \frac{c_0^2}{8\sqrt{n}} \epsilon^{-n+1}. \quad (3.11)$$

Indeed, (3.10) follows from the definition of A_δ , while (3.11) follows by (3.9). It follows from (3.10) and the definition of \hat{A} that

$$|Q'_\alpha \cap A| \geq |Q'_\alpha \cap A_1| > \frac{1}{2}|Q'_\alpha| \quad \text{and} \quad |(Q'_\alpha + z) \cap A^c| \geq |(Q'_\alpha + z) \cap \tilde{A}_2^c| > \frac{1}{2}|Q'_\alpha|.$$

Since A_1 does not intersect $(0, 1)^n \setminus (\epsilon_*, 1 - \epsilon_*)^n$, if $\epsilon < \epsilon^*/\sqrt{n}$ we obtain that $Q'_\alpha \subset (0, 1)^n$. Analogously, since A_2 contains $(0, 1)^n \setminus (\epsilon_*, 1 - \epsilon_*)^n$, if $\epsilon < \epsilon^*/\sqrt{n}$ we obtain that $Q'_\alpha + z \cap \partial Q = \emptyset$, which implies that there exists a vector h in \mathbb{Z}^n such that $Q'_\alpha + z + h \subset (0, 1)^n$ (to be precise, h is of the form $-\gamma_1 e_1 + \dots - \gamma_n e_n$ with $\gamma_i \in \{0, 1\}$).

Hence, by a continuity argument there exists $t_\alpha \in (0, 1)$ such that, setting $Q_\alpha := Q'_\alpha + t_\alpha(z + h)$, one has $Q_\alpha \subset (0, 1)^n$ and $|Q_\alpha \cap A| = |Q'_\alpha|/2$ (see Figure 2, that corresponds to the case $h = 0$). Then we can define \mathcal{G}_ϵ as the collection of the cubes Q_α , which is disjoint by construction (since their projections on H_a are disjoint). \square

We can now prove that $\liminf_{\epsilon} \mathbf{l}_{\epsilon}(f) \geq 1/2$. Set $c_0 = 2^{-n-1}$, let c_1 be given by Lemma 3.3, and set $K := 1/c_1$. If $\delta = \delta_0(A, K)$ is given by Lemma 3.2, we can apply Lemma 3.2 to obtain a finite disjoint family \mathcal{U}_{δ} of δ -cubes with $\#\mathcal{U}_{\delta} > c_1^{-1}\delta^{1-n}$ and

$$c_0 < \frac{|Q' \cap A|}{|Q'|} < 1 - c_0 \quad \text{for all } Q' \in \mathcal{U}_{\delta}.$$

Since \mathcal{U}_{δ} is finite, for $0 < \epsilon \ll \delta$ and all $Q' \in \mathcal{U}_{\delta}$ we can apply Lemma 3.3 to a rescaled copy by a factor δ^{-1} of Q' and $A \cap Q'$ to obtain a disjoint family $\mathcal{G}_{\epsilon}(Q')$ of ϵ -cubes contained in Q' and satisfying

$$\frac{|V \cap A|}{|V|} = \frac{1}{2} \quad \forall V \in \mathcal{G}_{\epsilon}(Q'), \quad (3.12)$$

$$\#\mathcal{G}_{\epsilon}(Q') > c_1 \left(\frac{\delta}{\epsilon} \right)^{n-1}. \quad (3.13)$$

Now, by construction, the family

$$\mathcal{G}_{\epsilon} := \bigcup_{Q' \in \mathcal{U}_{\delta}} \mathcal{G}_{\epsilon}(Q')$$

of ϵ -cubes is disjoint (taking into account condition (c) of Lemma 3.2) and $|V \cap A|/|V| = 1/2$ for each V in the family. In addition, its cardinality can be estimated from below as follows:

$$\#\mathcal{G}_{\epsilon} \geq c_1 \left(\frac{\delta}{\epsilon} \right)^{n-1} \#\mathcal{U}_{\delta} > c_1 \left(\frac{\delta}{\epsilon} \right)^{n-1} \frac{1}{c_1} \delta^{1-n} = \epsilon^{1-n}.$$

Extracting from \mathcal{G}_{ϵ} a subfamily \mathcal{F}_{ϵ} with $\#\mathcal{F}_{\epsilon} = \lceil \epsilon^{1-n} \rceil$ we get

$$\mathbf{l}_{\epsilon}(f) \geq \epsilon^{1-n} \sum_{V \in \mathcal{F}_{\epsilon}} \int_V \int_V |f(x) - f(y)| dx dy = 2\epsilon^{n-1} \sum_{V \in \mathcal{F}_{\epsilon}} \frac{|V \cap A| \cdot |V \setminus A|}{|V|^2} = \frac{1}{2} \epsilon^{n-1} \lceil \epsilon^{1-n} \rceil.$$

By taking the limit as $\epsilon \rightarrow 0^+$ the conclusion is achieved.

3.3. Lower bound: the rectifiable case. The heuristic idea of the proof is to choose cubes well adapted to the local geometry of ∂A , as in Figure 3 below. Although it is easy to make this argument rigorous if ∂A is smooth, when A has merely finite perimeter the argument becomes much less obvious. Still, the rectifiability of $\partial^* A$ and a suitable localization/blow-up argument allow us to prove the result in this general setting.

Let $A \subset \mathbb{R}^n$ be measurable and $\Omega \subset \mathbb{R}^n$ open. We localize $\mathbf{l}_{\epsilon}(\mathbf{1}_A)$ to Ω and, at the same time, we impose a scale-invariant bound on the cardinality of the families by defining

$$\mathbf{J}_{\epsilon}(A, \Omega) := \epsilon^{n-1} \sup_{\mathcal{F}_{\epsilon}} \sum_{Q' \in \mathcal{F}_{\epsilon}} \int_{Q'} \left| \mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A \right| dx,$$

where the supremum runs, this time, among all collections of disjoint families of ϵ -cubes contained in Ω , with arbitrary orientation and cardinality not exceeding $\mathbf{P}(A, \Omega)\epsilon^{1-n}$.

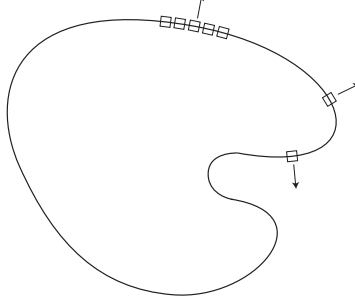


FIGURE 3.

Notice that J_ϵ has a nice scaling property, namely

$$J_\epsilon(A/r, \Omega) = r^{1-n} J_{r\epsilon}(A, r\Omega) \quad \forall r > 0. \quad (3.14)$$

In addition, the additivity of $P(A, \cdot)$ shows that J_ϵ is superadditive, namely

$$J_\epsilon(A, \Omega_1 \cup \Omega_2) \geq J_\epsilon(A, \Omega_1) + J_\epsilon(A, \Omega_2) \quad \text{whenever } \Omega_1 \cap \Omega_2 = \emptyset. \quad (3.15)$$

Then, the lower bound

$$2 \liminf_{\epsilon} l_\epsilon(\mathbf{1}_A) \geq \min\{1, P(A)\} \quad (3.16)$$

is a direct consequence of Theorem 3.4 below, choosing $\Omega = \mathbb{R}^n$. Indeed, since $P(A) \leq 1$ implies $J_\epsilon(A, \mathbb{R}^n) \leq l_\epsilon(\mathbf{1}_A)$ we obtain (3.16) when $P(A) \leq 1$. If $P(A) > 1$, let $k = [P(A)] \geq 1$ be its integer part and split any disjoint family \mathcal{F}_ϵ of ϵ -cubes with maximal cardinality which enters in the definition of $J_\epsilon(A, \mathbb{R}^n)$ into k subfamilies with cardinality $[\epsilon^{1-n}]$ and a remainder subfamily of cardinality not exceeding $P(A)\epsilon^{1-n} - k[\epsilon^{1-n}] \leq P(A)\epsilon^{1-n} - k(\epsilon^{1-n} - 1)$. Since \mathcal{F}_ϵ is arbitrary, recalling (3.1) we see that

$$J_\epsilon(A, \mathbb{R}^n) \leq k l_\epsilon(\mathbf{1}_A) + \frac{1}{2}(P(A) - k) + k\epsilon^{n-1}.$$

Applying once more Theorem 3.4 with $\Omega = \mathbb{R}^n$ yields

$$2 \liminf_{\epsilon \rightarrow 0} l_\epsilon(\mathbf{1}_A) \geq \frac{P(A)}{k} - \frac{(P(A) - k)}{k} = 1 = \min\{1, P(A)\}$$

since $P(A) > 1$.

Theorem 3.4. *For any measurable set $A \subset \mathbb{R}^n$ with finite perimeter in Ω one has*

$$\lim_{\epsilon \rightarrow 0^+} J_\epsilon(A, \Omega) = \frac{1}{2}P(A, \Omega).$$

The proof of the upper bound $\limsup_{\epsilon} J_\epsilon(A, \Omega) \leq P(A, \Omega)/2$ can be obtained exactly as in §3.1, so we focus on the lower bound. To this aim, it will be convenient to introduce the function

$$J_-(A, \Omega) := \liminf_{\epsilon \rightarrow 0^+} J_\epsilon(A, \Omega).$$

Because of (3.14) we get

$$J_-(A/r, \Omega) = r^{1-n} J_-(A, r\Omega) \quad \forall r > 0. \quad (3.17)$$

In addition, the superadditivity of $J_\epsilon(A, \cdot)$ and of the \liminf give

$$J_-(A, \Omega_1 \cup \Omega_2) \geq J_-(A, \Omega_1) + J_-(A, \Omega_2) \quad \text{whenever } \Omega_1 \cap \Omega_2 = \emptyset. \quad (3.18)$$

In the first lemma we consider (local) subgraphs of C^1 functions.

Lemma 3.5. *Let E be the subgraph of a C^1 function in a neighbourhood of 0. Then*

$$\liminf_{r \rightarrow 0^+} J_-(E/r, B_1) \geq \frac{1}{2} \omega_{n-1}.$$

Proof. The proof is elementary, just choosing the canonical division in ϵ -cubes, if $B_r \cap \partial E$ is contained in a hyperplane for $r > 0$ small enough. In the general case we use the fact that E/r is bi-Lipschitz equivalent to a half-space in B_1 , with bi-Lipschitz constants converging to 1 as $r \downarrow 0$. \square

In the second lemma we provide a sort of modulus of continuity for $E \mapsto J_\epsilon(E, \Omega)$.

Lemma 3.6. *Let $E, F \subset \Omega$ be sets of finite perimeter in Ω . Then*

$$J_\epsilon(F, \Omega) \leq J_\epsilon(E, \Omega) + \frac{1}{2} \epsilon^{n-1} + \mathcal{H}^{n-1}((\partial^* F \Delta \partial^* E) \cap \Omega) \quad \forall \epsilon > 0. \quad (3.19)$$

Proof. The inequality $\min\{z, 1-z\} \leq 2z(1-z)$ in $[0, 1]$ combined with (2.2) yields the relative isoperimetric inequality

$$\min\{|L|, |Q' \setminus L|\} \leq \frac{\epsilon}{2} P(L, Q') \quad \text{for any } \epsilon\text{-cube } Q' \text{ with } L \subset Q'. \quad (3.20)$$

Let now \mathcal{F}_ϵ be a family of ϵ -cubes contained in Ω with cardinality less than $\epsilon^{1-n} P(F, \Omega)$. For any $Q' \in \mathcal{F}_\epsilon$, adding and subtracting $\mathbf{1}_E$ we have

$$\int_{Q'} \int_{Q'} |\mathbf{1}_F(x) - \mathbf{1}_F(y)| dx dy \leq \int_{Q'} \int_{Q'} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| dx dy + \epsilon^{-n} |Q' \cap (F \Delta E)|.$$

Analogously, adding and subtracting $\mathbf{1}_{Q' \setminus E}$ and using $\mathbf{1}_{E^c}(x) - \mathbf{1}_{E^c}(y) = \mathbf{1}_E(y) - \mathbf{1}_E(x)$, we have

$$\int_{Q'} \int_{Q'} |\mathbf{1}_F(x) - \mathbf{1}_F(y)| dx dy \leq \int_{Q'} \int_{Q'} |\mathbf{1}_E(x) - \mathbf{1}_E(y)| dx dy + \epsilon^{-n} |Q' \cap (F \Delta E^c)|.$$

Since $F \Delta E = \Omega \setminus (F \Delta E^c)$, we can apply (3.20) with $L = Q' \cap (F \Delta E)$, single out from \mathcal{F}_ϵ a maximal subfamily with cardinality less than $\epsilon^{1-n} P(E, \Omega)$, and use (3.1) and the definition of $J_\epsilon(E, \Omega)$ to get

$$\begin{aligned} \epsilon^{n-1} \sum_{Q' \in \mathcal{F}_\epsilon} \int_{Q'} \int_{Q'} |\mathbf{1}_F(x) - \mathbf{1}_F(y)| &\leq J_\epsilon(E, \Omega) + \frac{1}{2} \epsilon^{n-1} + \frac{1}{2} (P(F, \Omega) - P(E, \Omega))^+ \\ &\quad + \frac{1}{2} \sum_{Q' \in \mathcal{F}_\epsilon} P(F \Delta E, Q'). \end{aligned}$$

Then, we use the additivity of $P(F \Delta E, \cdot)$ and take the supremum in the left hand side to obtain

$$J_\epsilon(F, \Omega) \leq J_\epsilon(E, \Omega) + \frac{1}{2}\epsilon^{n-1} + \frac{1}{2}(P(F, \Omega) - P(E, \Omega))^+ + \frac{1}{2}P(F \Delta E, \Omega),$$

and we conclude using (2.4) and (2.6). \square

Notice that, in particular, the previous lemma gives

$$J_-(F, \Omega) \leq J_-(E, \Omega) + \mathcal{H}^{n-1}((\partial^* F \Delta \partial^* E) \cap \Omega). \quad (3.21)$$

In the third lemma we prove a density lower bound for $J_-(E, \cdot)$ by comparing E on small scales with the subgraph of a C^1 function.

Lemma 3.7. *If E has locally finite perimeter in Ω , then*

$$\liminf_{r \rightarrow 0^+} \frac{J_-(E, B_r(x))}{\omega_{n-1} r^{n-1}} \geq \frac{1}{2} \quad \text{for } |D\mathbf{1}_E| \text{-a.e. } x \in \Omega. \quad (3.22)$$

Proof. Recall that $|D\mathbf{1}_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$ on Borel sets of Ω . In view of (3.21), the scaling property (3.17) of J_- , and Lemma 3.5, it suffices to show that for \mathcal{H}^{n-1} -a.e. $x \in \partial^* E$ there exists a set F which is the subgraph of a C^1 function in the neighbourhood of x , with

$$\mathcal{H}^{n-1}((\partial^* F \Delta \partial^* E) \cap B_r(x)) = o(r^{n-1}). \quad (3.23)$$

To this aim, we use the representation (2.5), we fix i and consider a point $x \in \Gamma_i \cap \partial^* E$ where²

$$\mathcal{H}^{n-1}((\Gamma_i \setminus \partial^* E) \cap B_r(x)) = o(r^{n-1}) \quad \text{and} \quad \mathcal{H}^{n-1}((\partial^* E \setminus \Gamma_i) \cap B_r(x)) = o(r^{n-1}).$$

In this way we obtain that (3.23) holds for \mathcal{H}^{n-1} -a.e. $x \in \Gamma_i \cap \partial^* E$ and the statement is proved, since i is arbitrary and Γ_i is a C^1 hypersurface. \square

We can now prove the missing part $J_-(E, \Omega) \geq P(E, \Omega)/2$ of Theorem 3.4. Let $S \subset \Omega$ be the set where the \liminf in (3.22) is greater or equal than $1/2$, and notice that Lemma 3.7 shows that $|D\mathbf{1}_E| \llcorner \Omega$ is concentrated on S , so that $\mathcal{H}^{n-1}(S) \geq P(E, \Omega)$. If $2J_-(E, \cdot) =: \mu(\cdot)$ were a σ -additive measure, then the well-known implication

$$\liminf_{r \rightarrow 0^+} \frac{\mu(B_r(x))}{\omega_{n-1} r^{n-1}} \geq 1 \quad \forall x \in S \implies \mu(\Omega) \geq \mathcal{H}^{n-1}(S) \quad (3.24)$$

would provide us with the needed inequality (see for instance [1, Theorem 2.56] for a proof of (3.24)). However, the traditional proof of (3.24) works also when μ is only a superadditive set function defined on open sets, as we illustrate below. In particular (3.24) is applicable to $2J_-(E, \cdot)$ in view of (3.18), which concludes the proof of Theorem 3.4. \square

²Here we use (first with $S = \Gamma_i$, then with $S = \partial^* E$) the property that $\mathcal{H}^{n-1}(S \cap B_r(x)) = o(r^{n-1})$ for \mathcal{H}^{n-1} -a.e. $x \in \mathbb{R}^n \setminus S$ whenever S has locally finite \mathcal{H}^{n-1} -measure, see for instance [1, pag. 79, Eq. (2.41)].

We now give a sketch of proof of (3.24) in the superadditive case, writing $k = n - 1$ for convenience. We can assume without loss of generality $S \subseteq \Omega$. To prove (3.24) we fix $\delta \in (0, 1)$ and consider all the open balls \mathcal{C} centered at points of L and with diameter $d_{\mathcal{C}}$ strictly less than δ , such that $\mu(\mathcal{C}) \geq (1 - \delta)\omega_k d_{\mathcal{C}}^k/2^k$. By applying Besicovitch covering theorem (see for instance [1, Theorem 2.17]) we obtain families $\mathcal{F}_1, \dots, \mathcal{F}_{\xi}$ (with $\xi = \xi(n)$ dimensional constant) with the following properties:

- (a) each family \mathcal{F}_i , $1 \leq i \leq \xi$, is disjoint;
- (b) $\cup_1^{\xi} \mathcal{F}_i$ contains S .

In particular, using the superadditivity of μ we can estimate from above the pre-Hausdorff measure $\mathcal{H}_{\delta}^k(L)$ as follows:

$$\mathcal{H}_{\delta}^k(S) \leq \sum_{i=1}^{\xi} \sum_{\mathcal{C} \in \mathcal{F}_i} \frac{\omega_k}{2^k} d_{\mathcal{C}}^k \leq \frac{1}{1 - \delta} \sum_{i=1}^{\xi} \sum_{\mathcal{C} \in \mathcal{F}_i} \mu(\mathcal{C}) \leq \frac{\xi}{1 - \delta} \mu(\Omega).$$

By letting $\delta \downarrow 0$ we obtain that $\mathcal{H}^k(S) \leq \xi \mu(\Omega) < \infty$. Using this information we can improve the estimate, now applying Besicovitch–Vitali covering theorem to the above mentioned fine cover of S , to obtain a disjoint family $\{\mathcal{C}_i\}$ which covers \mathcal{H}^k -almost all (hence \mathcal{H}_{δ}^k -almost all) of S . As a consequence

$$\mathcal{H}_{\delta}^k(S) \leq \sum_i \frac{\omega_k}{2^k} d_{\mathcal{C}_i}^k \leq \sum_i \frac{1}{1 - \delta} \mu(\mathcal{C}_i) \leq \frac{1}{1 - \delta} \mu(\Omega).$$

Letting $\delta \downarrow 0$ we finally obtain $\mathcal{H}^k(S) \leq \mu(\Omega)$, as desired. \square

3.4. Proof of Theorem 1.1 in the case $n = 1$. Note that

$$l_{\epsilon}(\mathbf{1}_A) := \sup_I \int_I |\mathbf{1}_A(x) - \int_I \mathbf{1}_A| dx, \quad (3.25)$$

and I runs among all intervals with length ϵ . Recall (see for instance [1]) that in the 1-dimensional case any set of finite and positive perimeter is equivalent to a finite disjoint union of closed intervals or half-lines, and the perimeter is the number of the endpoints; in addition $P(A) = 0$ if and only if either $|A| = 0$ or $|\mathbb{R} \setminus A| = 0$.

The inequalities $l_{\epsilon}(\mathbf{1}_A) \leq 1/2$ and $I_{\epsilon}(\mathbf{1}_A) \leq P(A)$ follow by (3.1) and (2.2) respectively, as in the case $n > 1$, hence $2 \limsup_{\epsilon} l_{\epsilon}(\mathbf{1}_A) \leq \min\{1, P(A)\}$. It remains to prove

$$2 \liminf_{\epsilon \rightarrow 0} l_{\epsilon}(\mathbf{1}_A) \geq \min\{1, P(A)\}$$

and, since $P(A)$ is always a natural number (possibly infinite), we need only to show that $P(A) \geq 1$ implies $2 \liminf_{\epsilon} l_{\epsilon}(\mathbf{1}_A) \geq 1$. We will prove the stronger implication

$$P(A) > 0 \implies 2 \liminf_{\epsilon \rightarrow 0} l_{\epsilon}(\mathbf{1}_A) \geq 1. \quad (3.26)$$

To prove (3.26), notice that $P(A) > 0$ implies that both A and $\mathbb{R} \setminus A$ have nontrivial measure, so there exist distinct points $x_1, x_2 \in \mathbb{R}$ such that x_1 is a density point of A and

x_2 is a density point of $\mathbb{R} \setminus A$. Hence, for ϵ sufficiently small we have then

$$\int_{(x_1-\epsilon/2, x_1+\epsilon/2)} \mathbf{1}_A > \frac{1}{2} \quad \text{and} \quad \int_{(x_2-\epsilon/2, x_2+\epsilon/2)} \mathbf{1}_A < \frac{1}{2}$$

We can then use a continuity argument to find, for ϵ sufficiently small, a point x_ϵ such that

$$\int_{(x_\epsilon-\epsilon/2, x_\epsilon+\epsilon/2)} \mathbf{1}_A = \frac{1}{2},$$

so that

$$\int_{(x_\epsilon-\epsilon/2, x_\epsilon+\epsilon/2)} \left| \mathbf{1}_A(x) - \int_{(x_\epsilon-\epsilon/2, x_\epsilon+\epsilon/2)} \mathbf{1}_A \right| dx = \frac{1}{2}.$$

Hence $P(A) > 0$ implies $2l_\epsilon(\mathbf{1}_A) \geq 1$ for ϵ small enough, so in particular $2 \liminf_\epsilon l_\epsilon(\mathbf{1}_A) \geq 1$, as desired. \square

4. VARIANTS

4.1. A localized version of Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary and consider the quantity

$$l_\epsilon(f, \Omega) := \epsilon^{n-1} \sup_{\mathcal{F}_\epsilon} \sum_{Q' \in \mathcal{F}_\epsilon} \int_{Q'} |f(x) - \int_{Q'} f| dx, \quad (4.1)$$

where \mathcal{F}_ϵ denotes a collection of disjoint ϵ -cubes $Q' \subset \Omega$ with *arbitrary* orientation and cardinality not exceeding ϵ^{1-n} .

In analogy with Theorem 1.1, we can also prove the following result.

Theorem 4.1. *For any measurable set $A \subset \mathbb{R}^n$ one has*

$$\lim_{\epsilon \rightarrow 0} l_\epsilon(\mathbf{1}_A, \Omega) = \frac{1}{2} \min\{1, P(A, \Omega)\}.$$

Proof. We begin by noticing that both the upper and the lower bound in the rectifiable case are local, so that parts of the proof go throughout without any essential modification. Hence, we only need to discuss the lower bound in the non-rectifiable case.

If $P(A, \Omega) = \infty$, we can find an open smooth subset $\Omega' \Subset \Omega$ such that $P(A, \Omega')$ is arbitrarily large (the largeness will be fixed later). We set $c_0 = 2^{-n-1}$, consider c_1 be given by Lemma 3.3, and set $K := 1/c_1$. Then, by looking at the proof of Lemma 3.2 it is immediate to check that the same result still holds with $K = 1/c_1$ and considering only δ -cubes which intersect Ω' (this ensures that, if δ is sufficiently small, all cubes are contained inside Ω) provided $P(A; \Omega') > 2^{2n+2}nK$. Thanks to this fact, the proof at the end of Section 3.2 now goes through without modifications: first we apply Lemma 3.2 to find a disjoint family \mathcal{U}_δ of δ -cubes intersecting Ω' with $\#\mathcal{U}_\delta > c_1^{-1}\delta^{1-n}$ and

$$c_0 < \frac{|Q' \cap A|}{|Q'|} < 1 - c_0 \quad \text{for all } Q' \in \mathcal{U}_\delta,$$

and then we apply Lemma 3.3 with $\epsilon \ll \delta$ to obtain, for each $Q' \in \mathcal{U}_\delta$, a disjoint family $\mathcal{G}_\epsilon(Q')$ of ϵ -cubes satisfying

$$\frac{|V \cap A|}{|V|} = \frac{1}{2} \quad \forall V \in \mathcal{G}_\epsilon(Q'),$$

$$\#\mathcal{G}_\epsilon(Q') > c_1 \left(\frac{\delta}{\epsilon} \right)^{n-1}.$$

We then conclude as in Section 3.2. \square

Next, we return to the quantity $[f]$ defined in [6]. As announced in the introduction, we establish the following result:

Corollary 4.2. *For any measurable set $A \subset Q$ one has*

$$[\mathbf{1}_A] \leq \frac{1}{2} \min\{1, \mathbf{P}(A, Q)\} \leq C [\mathbf{1}_A].$$

Proof. The result is an immediate consequence of Theorem 4.1 and Lemma 4.3 below, where we compare the quantities $\mathbf{l}_\epsilon(f)$ with their anisotropic counterparts $[f]_\epsilon$ as defined in [6]. \square

Lemma 4.3. *For $\epsilon > 0$ and $f \in L^1(Q)$ measurable, let $\mathbf{l}_\epsilon(f, Q)$ be defined as in (1.7) with cubes $Q' \subset Q$ and let $[f]_\epsilon$ be defined by (1.1). Then*

$$[f]_\epsilon \leq \mathbf{l}_\epsilon(f, Q) \leq C [f]_{\sqrt{n}\epsilon} \quad \forall \epsilon \in (0, 1/\sqrt{n}). \quad (4.2)$$

In the proof of this result, we shall to use the elementary inequalities

$$\int_Q |f(x) - \int_Q f| dx \leq \int_Q \int_Q |f(x) - f(y)| dx dy \leq 2 \int_Q |f(x) - \int_Q f| dx. \quad (4.3)$$

Proof. The first inequality in (4.2) is obvious. In order to prove the second one, let $\mathcal{F}_\epsilon = \{Q_i\}_{i \in I}$ be a disjoint family of ϵ -cubes in Q with cardinality of I less than ϵ^{1-n} . For each cube Q_i in \mathcal{F}_ϵ we can find a $\sqrt{n}\epsilon$ -cube Q'_i containing Q_i , contained in Q , and with sides parallel to the coordinate axes. Since \mathcal{F}_ϵ is disjoint, the family of the corresponding cubes Q'_i has bounded overlap, more precisely for each cube Q'_i the cardinality of the set $\{j : Q'_j \cap Q'_i \neq \emptyset\}$ does not exceed c_n . Hence, by an exhaustion procedure, we can partition the index set I in families I_1, \dots, I_N , with $N \leq c_n$, in such a way that the families

$$\mathcal{G}'_j := \{Q'_i : i \in I_j\} \quad j = 1, \dots, N$$

are disjoint. Since $\#\mathcal{G}'_j \leq \epsilon^{1-n}$ for each $j = 1, \dots, N$, splitting the family \mathcal{G}'_j in at most $n^{(n-1)/2} + 1$ subfamilies with cardinality less than $(\sqrt{n}\epsilon)^{1-n}$, we have

$$(\sqrt{n}\epsilon)^{1-n} \sum_{Q' \in \mathcal{G}'_j} \int_{Q'} |f(x) - \int_{Q'} f| dx \leq (n^{(n-1)/2} + 1) [f]_{\sqrt{n}\epsilon}.$$

On the other hand, if \mathcal{G}_j , $j = 1, \dots, N$, denote the corresponding families of original cubes, since

$$\int_Q \int_Q |f(x) - f(y)| dx dy \leq (\sqrt{n})^{2n} \int_{Q'} \int_{Q'} |f(x) - f(y)| dx dy,$$

using (4.3) we readily obtain

$$\sum_{Q \in \mathcal{G}_j} \int_Q \left| f(x) - \int_Q f \right| dx \leq 2n^n \sum_{Q' \in \mathcal{G}'_j} \int_{Q'} \left| f(x) - \int_{Q'} f \right| dx.$$

Hence, since $\mathcal{F}_\epsilon = \mathcal{G}_1 \cup \dots \cup \mathcal{G}_N$, adding with respect to j and using the fact that \mathcal{F}_ϵ is arbitrary, we obtain the second inequality in (4.2) with

$$C := 2c_n(n^{(n-1)/2} + 1)n^{(3n-1)/2}.$$

□

Remark 4.4. Notice that Corollary 4.2 could be refined by adapting the argument used in Section 3.3 to the setting of [6] where the cubes are forced to be parallel to the coordinate axes: more precisely, if we denote by K_n the largest possible intersection of a hyperplane with the unit cube, i.e.,

$$K_n := \sup_{H \subset \mathbb{R}^n \text{ hyperplane}} \mathcal{H}^{n-1}(H \cap Q),$$

then

$$[\mathbf{1}_A] \geq \frac{1}{2} \min \left\{ 1, \frac{1}{K_n} \mathbf{P}(A, Q) \right\}.$$

4.2. A new characterization of the perimeter. Theorem 1.1 provides a characterization of sets of finite perimeter only when $\lim_{\epsilon \rightarrow 0} \mathbf{l}_\epsilon(\mathbf{1}_A) < 1/2$. This critical threshold could be easily tuned by modifying the upper bound on the cardinality of the families \mathcal{F}_ϵ in (1.7), as (1.9) shows. As a consequence a byproduct of our results is a characterization of the perimeter free of truncations:

$$\limsup_{\epsilon \downarrow 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| \mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A \right| dx = \mathbf{P}(A), \quad (4.4)$$

where now \mathcal{H}_ϵ denotes a collection of disjoint ϵ -cubes $Q' \subset \mathbb{R}^n$ with arbitrary orientation but no constraint on cardinality.

In order to prove (4.4) we notice that the argument in Section 3.1, based on the relative isoperimetric inequality, easily gives

$$\epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| \mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A \right| dx \leq \mathbf{P}(A).$$

On the other hand, we can use (1.9) to get

$$\liminf_{\epsilon \downarrow 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| \mathbf{1}_A(x) - \int_{Q'} \mathbf{1}_A \right| dx \geq \sup_{M > 0} \min\{M, P(A)\} = P(A),$$

proving (4.4).

Notice that the formulation given in Theorem 1.1 is stronger than (4.4), because it shows that a cardinality constrained maximization is sufficient to provide finiteness of perimeter, under the critical threshold.

It is also worth noticing that (4.4) can be extended to general \mathbb{Z} -valued functions: indeed, the argument in Section 3.3 can be easily adapted to prove that if $f \in BV(\mathbb{R}^n; \mathbb{Z})$ then

$$\liminf_{r \rightarrow 0} \frac{J_-(f; B_r(x))}{|Df|(B_r(x))} \geq \frac{1}{2} \quad \text{for } |Df|\text{-a.e. } x,$$

where $J_-(f; B_r(x))$ is defined analogously to $J_-(E; B_r(x))$ in Section 3.3, thus showing that

$$\liminf_{\epsilon \downarrow 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| f(x) - \int_{Q'} f \right| dx \geq |Df|(\mathbb{R}^n),$$

while the converse inequality follows by writing

$$f = \sum_{k > 0} \mathbf{1}_{\{f \geq k\}} - \sum_{k > 0} \mathbf{1}_{\{f \leq -k\}},$$

which gives

$$\begin{aligned} & \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| f(x) - \int_{Q'} f \right| dx \\ & \leq \sum_{k > 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| \mathbf{1}_{\{f \geq k\}}(x) - \int_{Q'} \mathbf{1}_{\{f \geq k\}} \right| dx \\ & \quad + \sum_{k > 0} \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} \left| \mathbf{1}_{\{f \leq -k\}}(x) - \int_{Q'} \mathbf{1}_{\{f \leq -k\}} \right| dx \\ & \leq \sum_{k > 0} P(\{f \geq k\}) + \sum_{k > 0} P(\{f \leq -k\}) = |Df|(\mathbb{R}^n). \end{aligned}$$

These facts provide a new characterization both of sets of finite perimeter and of the perimeter of sets, independent of the theory of distributions. Heuristically, given $\epsilon > 0$, any maximizing family \mathcal{F}_ϵ provides a sort of boundary on scale ϵ of A , and some proofs (in particular the one of Lemma 3.2, see also Remark 3.1) make more rigorous this idea.

It is interesting also to compare this result with another non-distributional characterization of sets of finite perimeter due to H. Federer, see [12, Theorem 4.5.11]: $P(A)$ is finite if and only if the essential boundary $\partial^* A$, namely the set in (2.3) of points of density neither 0 nor 1, has finite \mathcal{H}^{n-1} -measure, and then $P(A) = \mathcal{H}^{n-1}(\partial^* A)$. However, Federer's characterization and the one provided by this paper seem to be quite different.

4.3. Approximation of the total variation. Motivated by the results in Section 4.2, for $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ we may define

$$K_\epsilon(f) := \sup_{\mathcal{H}_\epsilon} \epsilon^{n-1} \sum_{Q' \in \mathcal{H}_\epsilon} 2 \int_{Q'} |f(x) - \int_{Q'} f| dx$$

where, once more, \mathcal{H}_ϵ denotes a collection of disjoint ϵ -cubes $Q' \subset \mathbb{R}^n$ with arbitrary orientation but no constraint on cardinality. Then, the result of the previous section can be read as follows:

$$\lim_{\epsilon \rightarrow 0} K_\epsilon(f) = |Df|(\mathbb{R}^n)$$

for any \mathbb{Z} -valued function f .

For general BV_{loc} functions f , the asymptotic analysis of K_ϵ seems to be more difficult to grasp. By considering smooth functions and functions with a jump discontinuity along a hyperplane, one is led to the conjecture that

$$\lim_{\epsilon \rightarrow 0} K_\epsilon(f) = \frac{1}{4} |D^a f|(\mathbb{R}^n) + \frac{1}{2} |D^s f|(\mathbb{R}^n) \quad (4.5)$$

for all $f \in SBV_{\text{loc}}(\mathbb{R}^n)$, where (see [1]) $SBV_{\text{loc}}(\Omega)$ is the vector space of all $f \in BV_{\text{loc}}(\Omega)$ whose distributional derivative is the sum of a measure $D^a f$ absolutely continuous w.r.t. \mathcal{L}^n and a measure $D^s f$ concentrated on a set σ -finite w.r.t. \mathcal{H}^{n-1} . For functions $f \in BV_{\text{loc}} \setminus SBV_{\text{loc}}$, having the so-called Cantor part of the derivative, it might possibly happen that $K_\epsilon(f)$ oscillates as $\epsilon \rightarrow 0$ between $\frac{1}{4} |Df|(\mathbb{R}^n)$ and $\frac{1}{2} |Df|(\mathbb{R}^n)$.

Notice that all functionals K_ϵ are $L^1_{\text{loc}}(\mathbb{R}^n)$ -lower semicontinuous. On the other hand, it is natural to expect that the Γ -limit of K_ϵ w.r.t. the $L^1_{\text{loc}}(\mathbb{R}^n)$ topology exists and that

$$\Gamma - \lim_{\epsilon \rightarrow 0} K_\epsilon(f) = \frac{1}{4} |Df|(\mathbb{R}^n).$$

5. APPENDIX: PROOF OF (2.2)

In this section we prove the relative isoperimetric inequality in the cube, in the sharp form provided by (2.2), in any Euclidean space \mathbb{R}^n , $n \geq 1$. To this aim, we introduce the Gaussian isoperimetric function $I : (0, 1) \rightarrow (0, 1/\sqrt{2\pi}]$ defined by

$$I(t) := \varphi \circ \Phi^{-1}(t) \quad \text{with} \quad \varphi(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad \Phi(x) := \int_{-\infty}^x \varphi(y) dy, \quad x \in \mathbb{R}.$$

We extend I by continuity to $[0, 1]$ setting $I(0) = I(1) = 0$. Notice that $I(1/2) = \varphi(0) = 1/\sqrt{2\pi}$ and it is also easy to check that $I(t) = I(1-t)$.

Lemma 5.1. *The function $K(t) := \sqrt{2\pi}I(t) - 4t(1-t)$ is nonnegative in $[0, 1]$ and $K(t) = 0$ if and only if $t \in \{0, 1/2, 1\}$.*

Proof. Let us record an additional property of I :

$$I \in C^\infty(0, 1) \quad \text{and} \quad I'' = -\frac{1}{I} \quad \text{on } (0, 1). \quad (5.1)$$

Indeed, from $I \circ \Phi = \varphi$ and $\Phi' = \phi$ we obtain $(I' \circ \Phi(x))\varphi(x) = \varphi'(x) = -x\varphi(x)$, so that $I' \circ \Phi(x) = -x$. By differentiating once more we get $I''(t) = -1/\Phi' \circ \Phi^{-1}(t) = -1/I(t)$ in $(0, 1)$, as desired.

Since I attains its maximum at $1/2$, from (5.1) we obtain that $I' > 0$ in $(0, 1/2)$, hence there exists a unique $t_0 \in (0, 1/2)$ such that $I(t_0) = \sqrt{2\pi}/8$.

Since K vanishes on $\{0, 1/2, 1\}$ and it inherits from I the symmetry property $K(t) = K(1-t)$, it suffices to check that K is strictly positive in $(0, 1/2)$. Differentiating K we get

$$K'(t) = \sqrt{2\pi}I'(t) - 4 + 8t, \quad (5.2)$$

In particular, $K'(1/2) = I'(1/2) = 0$. Differentiating once more and using (5.1) we get

$$K''(t) = \sqrt{2\pi}I''(t) + 8 = -\frac{\sqrt{2\pi}}{I(t)} + 8, \quad (5.3)$$

so that

$$K'' < 0 \quad \text{in } (0, t_0), \quad K'' > 0 \quad \text{in } (t_0, 1/2). \quad (5.4)$$

In particular $K'(1/2) = 0$ gives $K' < 0$ in $[t_0, 1/2)$ and therefore $K(1/2) = 0$ gives

$$K > 0 \quad \text{in } [t_0, 1/2). \quad (5.5)$$

For the interval $[0, t_0]$ we use $K(t_0) > 0$, $K(0) = 0$ and the concavity of K in $(0, t_0)$, ensured by (5.4), to get

$$K > 0 \quad \text{in } (0, t_0]. \quad (5.6)$$

The conclusion follows by (5.5) and (5.6). \square

Now, let us prove (2.2). Combining [3, Proposition 5 and Theorem 7] we obtain the inequality

$$I\left(\int_{(0,1)^n} f \, dx\right) \leq \int_{(0,1)^n} \left[I(f) + \frac{1}{\sqrt{2\pi}}|\nabla f|\right] dx \quad (5.7)$$

for any locally Lipschitz function $f : (0, 1)^n \rightarrow [0, 1]$. Then, the BV version of the Meyers-Serrin approximation theorem (due to Anzellotti-Giaquinta, see for instance [1, Theorem 3.9]) enables us to approximate in $L^1((0, 1)^n)$ any function $f \in BV((0, 1)^n)$ by functions $f_k \in C^\infty((0, 1)^n)$ in such a way that

$$\lim_{k \rightarrow \infty} \int_{(0,1)^n} |\nabla f_k| \, dx = |Df|((0, 1)^n).$$

In addition, if $f : (0, 1)^n \rightarrow [0, 1]$, a simple truncation argument provides approximating functions f_k with the same property. It then follows from (5.7) that

$$I\left(\int_{(0,1)^n} f \, dx\right) \leq \int_{(0,1)^n} I(f) \, dx + \frac{1}{\sqrt{2\pi}}|Df|((0, 1)^n)$$

for all $f : (0, 1)^n \rightarrow [0, 1]$ with bounded variation. Since $I(0) = I(1) = 0$, choosing $f = \mathbf{1}_E$ gives

$$I(|E|) \leq \frac{1}{\sqrt{2\pi}} P(E, (0, 1)^n).$$

We conclude using Lemma 5.1.

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